Voting power and fair representation: Back to basics

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Abstract

This paper is an excursion into the sort of powers conferred by voting institutions. Its central argument is that inclusion as a member of a committee system spins a web of such *a priori powers* leaving no unambiguous answer to questions such as: were the countries in the initial configurations of the European Council of Ministers fairly represented? As we will see, decision-making bodies grant distinct if related abilities – the power to determine one's own success, the vulnerability to or immunity from other members' powers – whose recognition yields a richer analysis of voting institutions. Some of these abilities have a history of measurement going back more than half a century. The task before us in the balance of the text will be to supplement these indices with positive measures of the remaining powers obtaining between voting members.

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> 'Full and effective participation by all citizens in state government requires [...] that each citizen have an equally effective voice in the election of members of his state legislature. Modern and viable state government needs, and the Constitution demands, no less.'
>
> Chief Justice Earl Warren[1]

Of all the social revolutions in the fractured times of the 1960s, the reapportionment upheaval rarely bears mention. Yet, the short space of two years between 1963 and 1964 is packed with significance for the attainment of that 'inalienable right' to 'fair and effective [political] participation'[2] guaranteed by the equal protection clause of the US Constitution. During that time, a series of cases decided by the Warren Supreme Court—and later dubbed its 'success story'[3]—cut through the 'political thicket'[4] of legislative reapportionment to arrive at the now famous 'one person, one vote' principle.[5] That is, the standard of political equality that 'one man's vote [...] is to be worth as much as another's'.[6]

The present paper is concerned with the burgeoning literature born out of these reapportionment decisions, i.e. the mathematical theory of measuring *a priori* voting power. Half a century separates us from the original contributions of American lawyer and mathematician John F. Banzhaf III to the practical implementation of the 'one person, one vote' principle[7] – a period during which their domain has shifted from achieving equality in a representative democracy[8] to the more general problem of gauging the voting power of participants in any decision-making body.

1 Opinion of the US Supreme Court in *Reynolds v. Sims*, 377 U.S. 533 (1964), at 565.

2 Ibid.

3 McKay (1968).

4 A phrase due to Justice Frankfurter in *Colegrove v. Green*, 328 U.S. 549 (1946), at 556.

5 Coined by Justice Douglas in *Gray v. Sanders*, 372 U.S. 368 (1963), at 381, and often invoked as the 'one man, one vote' principle. The phrase is currently causing a rekindling of the reapportionment revolution in the case of *Evenwel v. Abbott* where the Supreme Court would have to decide whether 'person' refers to—and, hence, representation is based on—the total population of a district or its qualified voters only.

6 Opinion of the Court in *Wesberry v. Sanders*, 376 U.S. 1 (1964), at 8.

7 There are, in fact, four independent fathers of the field: the Maryland delegate to the 1787 Philadelphia Convention, Luther Martin (1787); the geneticist and mathematician Lionel Penrose (1946, 1952); John F. Banzhaf III (1965, 1966, 1968a, 1968b); and the mathematician and game theorist Lloyd S. Shapley (1953, together with Martin Shubik 1954). The antecedence of the former two, however, has gained recognition only relatively recently – see Felsenthal and Machover (2005, 1998: 6–10), Grofman and Scarrow (1979), Morris (1987: 182–190) and Riker (1986) for partial historical accounts.

8 Of the Court's insistence on equalising voting power, as Dixon (1964: 210) points out, '[f]air representation is the ultimate goal'. Yet, 'equal representation does not necessarily mean fair representation' (Carpeneti 1972: 669) – issues such as gerrymandering and the choice of voting procedure could debase the vote of entire groups. See Engstrom (1976) and Dixon (1964, 1965, 1969) for a persistent critique of the fixation on mathematical equality.

Yet, despite this, the conceptual basis of the field has remained strikingly unchanged. To this day, effective participation or voting power – at least in an *a priori* context, a term that will be made more precise in the next section – largely means what it meant for Banzhaf: the ability of a voter to determine the success or failure of a bill. This has led to an ever intensifying critique, mostly by game theorists, contesting how useful such *a priori* measures are. The bone of contention is whether and how interactions *between* players should feature in power metrics. Strategic solution concepts, for example, capture relational aspects of the voting environment that traditional indices ignore. The drawback, however, is that by relying on voters' preferences, such solutions swerve away from measuring power proper. To wit, such metrics reveal much about what voters *would* but not *can* do.

After laying some conceptual groundwork in the first part, the rest of the paper describes in more detail and answers this critique by developing a number of purely *a priori* relational indices. To anticipate, the aim of the measures in section three is to suggest that there is indeed no *single* ability called *a priori* power, such as Banzhaf's understanding of an effective vote. There is, in fact, a plurality of power ascriptions that can be made in an *a priori* context and they not only enrich the conceptual and positive analysis of a decision-making body, but could lead to diverging evaluations of how fair such committees are. To illustrate, the final section concludes with an application to the European Union's Council of Ministers. As we will see, how fair a representative democratic institution, such as the Council, is depends substantially on what type of power one looks at. Under-representation according to one could morph into over-representation according to another.

1 Power, voting power and *a priori* power

The study of power in philosophy, political science and sociology presents a curious case of Whig history. To wit, for all the past debates on the difference between outcome ('power to') and relational ('power over') power,[9] there is today a tight consensus around a 'generic concept'[10] of the term: the ability or capacity to effect an outcome. That is, at least since the 1980s, 'power over' has been recognised as 'parasitic upon'[11] 'power to', the former being obtainable by a mere re-description

[9] General discussions can be found in Dowding (1991: 48), Morriss (1987: 32–35) and Isaac (1987). A thorough, if despite this incomplete, conceptual history of the field is in Zimmerling (2005: 15–95).

[10] Braham (2008: 11). 'Generic' comes from Morriss' (1987: 48–51) distinction between generic *abilities* and time-specific *ableness*.

[11] Isaac (1987: 15).

of the relevant outcome.[12]

Power then, at its core, is a dispositional property. And conceding that '[e]verything that needs to be said about power can be said using the idea of the capacity to effect outcomes'[13] yields a strikingly useful two-part definition. Relational power, for instance, is simply an *ability* with respect to a relational *outcome*, while voting power is an *ability* with respect to a voting *outcome*. What could such a voting outcome be?

Banzhaf's answer in 1965 was 'the passage or defeat of a particular bill'[14] – a conception that, aided by Shapley and Shubik's earlier choice of 'the success of a winning coalition',[15] has carried over to the present day as the predominant characterisation of a voting outcome. While deeply intuitive – how a ballot turns out, after all, is of dear concern to a voter – it has led to an unfortunate and unfair critique – how a ballot turns out, however, need not be the *only* concern of a voter. To understand the thrust of this critique, let us revisit a common distinction between *a priori* and *a posteriori* measures of power.

A priori indices – in the simple context of yes-no voting – analyse 'the rules themselves'; that is, they 'abstract[s] away from the particular personalities and political interests present in particular voting environments'[16] in order to evaluate the design of voting rules, such as [the decision-making processes legislated by] constitutions or representation systems in national governments. As in such models the 'interests' and 'personalities' of voters are characterised by preference functions, abstracting them away amounts to employing only the resources of a *game form*, i.e. a set of players, strategies and corresponding outcomes (determined by a quota, for example).[17] But the inclusion of strategies and outcomes immediately implies an assumption about the players' behaviour.[18] This behaviour is summarised by the assumption of equiprobable distribution which gives the indices their *a priori* label. That is, the assumption that a 'yea' vote is as likely as a 'nay' vote, each with an independent probability one half.

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[12] The emergent consensus is largely due to Morriss (1987) who – within the space of three decades – managed to convert even the radical Lukes (1974, 2005).

[13] Morriss (1987: 34).

[14] Banzhaf (1965: 331).

[15] Shapley and Shubik (1954: 787).

[16] Roth (1988: 9).

[17] Game forms – games without preferences – are due to Gibbard (1973) in the development of his manipulability result, i.e. the Gibbard–Satterthwaite theorem. Miller (1982) is an application to *a priori* power measurement while Braham and Holler (2005a) is a conceptual argument for game forms in the same context.

[18] As Laruelle and Valenciano (2005: 181) note, for the '*descriptive* evaluation of a voting situation [. . .] the voting rule is not sufficient, an estimate of the voters' voting behavior is also needed'.

A posteriori measures, in contrast, gauge the actual distribution of power in a committee, taking into account all relevant – or at least potentially quantifiable – information about a voting environment such as players' preferences or positions on a political spectrum which drive coalitional formations, the type and probability of proposing various motions, special properties of the institutional process such as agenda-setting in the EU Council of Ministers, etc. Naturally, incorporating – at the very least – the 'personal, psychological differences' and 'affinities' among players[19] leads to abandoning the equiprobability assumption. So what is the justification of making it in the first place?

Felsenthal and Machover summarise the underlying argument as follows:

> First, taking any particular voter, *a*: if we have no information as to *a*'s personality and interests, or the nature of the bill to be voted upon, then we ought to assign equal *a priori* probabilities to *a* voting either way. Second, in ignorance of any concurrence or opposition of interests between voters, the most rational *a priori* assumption to adopt is that of independence.[20]

This, of course, is Rawls' *veil of ignorance* argument often invoked together with the *principle of insufficient reason* from probability theory. Unsurprisingly perhaps, the justification – as presently stated – has been heavily criticised. To wit, if we are indeed in a state of complete ignorance, why assume equiprobability rather than any other distribution? And if there is indeed scarce or ample evidence on voter affiliations, why not use it? An answer to the first question is that the argument, in fact, goes the other way around: it is *because a priori* we should consider every action equally possible that we assign equal probabilities (and impose the veil of ignorance).[21] A short answer to the second is that allowing for the actual preferences of voters might give an estimate of the power they have *exercised* but not the power they *posses*.[22]

A consequence of this division of labour – buttressed by the implicit association of 'power over' ascriptions with influence or coercive behaviour – has been the following

[19] Owen (1971: 346).

[20] Felsenthal and Machover (1998: 38).

[21] Which in the binary case amounts to one half. The argument is not only conceptual – that is, acknowledging that when measuring ability one should evaluate all possible actions and not just those one would rather do. When distributing power among representatives of states, for example, an institution should allow for governments of any shade on the spectrum and should allow them to vote even against that shade. In this sense, under this assumption, the abilities such a rule grants respect sovereignty and sovereign choice.

[22] That is, it amounts to committing Morriss' (1987: 15–18) *exercise fallacy*. For the longer answer see, in particular, Braham and Holler (2005a, 2005b) with responses in Napel and Widgrén (2005) and Schmidtchen and Steunenberg (2014).

conclusion. As relational power involves the preferences of voters and as preferences are the bread and butter of (the largely non-cooperative game theory based) *a posteriori* analysis, there can be no meaningful *a priori* 'power over' measures. In other words, classical *a priori* indices need to be enriched by strategic solution concepts – working with specific preference distributions – in order to elicit the interactions prevalent in a voting context. The balance of the text aims to counter this critique by developing a number of such relational *a priori* indices – the language of which is reviewed in the next section.

2 Core concepts and notation

A natural language for a legislative process with yes-no voting is that of *simple games*. Let N be the *grand coalition* of all relevant players, or voters, in a committee and let S be any subset – *coalition* – of N. Further, let v be the characteristic function, i.e. *worth*, of S. As a coalition can be either winning ($v(S) = 1$) or losing ($v(S) = 0$), such a voting game can be represented by its set of winning coalitions, $W(v)$. That is, a simple game is a pair $(N, W(v))$ satisfying the following properties: $N \in W(v)$, i.e. unanimous support guarantees winning, and $\varnothing \notin W(v)$, i.e. unanimous opposition guarantees losing. Two additional properties that are commonly imposed are propriety – for all $S \subseteq N$, if $S \in W(v)$, then $N \setminus S \notin W(v)$, i.e. if a coalition is winning, then its complement is losing – and monotonicity – if $S \in W(v)$ and $S \subseteq T \subseteq N$, then $T \in W(v)$, i.e. a coalition that contains a winning coalition is itself winning.[23]

A special class of simple games – that is particularly interesting for voting power analysis and that will be the main focus throughout the rest of the text – is the class of *weighted voting games* (WVG). A WVG is given by a vector of real non-negative numbers $(w_1, ..., w_n)$ – the players' voting *weights* – and a quota $q \in \left(0, \sum_{i \in N} w_i\right]$ – the committee's *decision rule* – that needs to be met for a coalition to qualify as winning.

Now, the key idea behind representing what it means to have power – an ability to determine an outcome – is that of a *swing*. A voter i has a swing if, given a formed coalition S, were he to change his vote, that would change the outcome. In other words, counterfactually, he has the ability to make a difference: $v(S) - v(S \setminus \{i\}) =$

[23] That is, additional support cannot hurt a coalition. See Shapley (1954) and Peleg and Sudhölter (2007: 16–18) for concise reviews and Felsenthal and Machover (1998) for an authoritative book-length presentation.

1. Intuitively, the swing is the formal equivalent of the first element in our two-part definition of power. Formalising a relevant outcome yields measures of various conceptions of interest.

As we have seen, the Banzhaf index, for example, – one of the two most prominent *a priori* indices[24] – specifies the passage ($v(S) = 1$) or defeat ($v(S) = 0$) of a bill. That is, a voter i has the power to determine the success or failure of a coalition S if: $[v(S) - v(S \setminus \{i\})] \cdot v(S) = 1$. A moment's reflection should convince you that in monotonic games the second argument is redundant. Summing over all possible coalitions then yields the Banzhaf value, $\eta_i(v)$, i.e. the number of coalitions in game v in which i has a swing with respect to the passage or defeat of a motion:

$$\eta_i(v) = \sum_{S \subseteq N} [v(S) - v(S \setminus \{i\})]$$

The (absolute) Banzhaf index of voter i, $\beta_i(v)$, is simply the ratio between the number of actual to potential such swings:[25]

$$\beta_i(v) = \frac{\eta_i(v)}{2^{n-1}}$$

where n is the number of all voters in the committee: $n = |N|$.[26] Because of the prominence of this kind of power in the literature, let us fix the term *decisiveness* for it. That is, a voter is decisive if he has (some) ability to effect the way the ballot turns out, or – equivalently, in the simple case of yes-no voting – if he has (some) ability to effect his own success.

Yet, remembering the two-part definition of power, it is clear that the success or failure of a bill is just one of a number of outcomes we could be interested in.[27] To wit, the critique launched by proponents of strategic indices probes into the very worry that important outcomes are indeed omitted – outcomes obtaining *between* voters and eliciting interesting interdependencies. Let us revisit the critique by way

[24] The second being the Shapley-Shubik index. Instead of on random combinations, the Shapley-Shubik measure is based on random permutations where the correlate of 'having a swing' is 'being pivotal'. The indices in the next section follow Banzhaf's framework but analogue results can be obtained by taking Shapley and Shubik's approach.

[25] This (cooperative) game theoretic version of $\beta_i(v)$ is due to Dubey and Shapley (1979).

[26] There are a total of 2^n possible coalitions with a committee of n voters. As yes and no votes are equiprobable, i belongs to – votes 'yea' in – exactly half of them.

[27] There are by now myriads of variations on Banzhaf's – and Shapley-Shubik's – theme. Coleman's (1971: 280–283) indices of a voter's power to *initiate* and to *prevent* a motion, for example, consider the passage or defeat of a motion as distinct outcomes. See also the Deegan-Packel (1978) measure and Holler and Packel's (1983) Public Good Index – the latter being a response to the former – as well as Johnston's (1978) index.

of a concrete example due to Napel and Widgrén (2001).[28]

Consider the WVG characterised by a quota $q = 5$ and three voters A, B, C with weights $(3, 2, 2)$. There are three winning coalitions – namely, $\{AB\}$, $\{AC\}$ and $\{ABC\}$ – in each of which voter A has a swing. B and C, on the other hand, are critical in only one coalition each. Calculating the Banzhaf index yields the following *a priori* power distribution: $(\frac{3}{4}, \frac{1}{4}, \frac{1}{4})$. This distribution, the critique goes, does not reflect fully A's power. To wit, as A holds the ultimatum card – she is a *veto* player – and no bill can be passed without her approval, surely A's power should be comparatively larger than the sorry portion of $\frac{3}{4}$. The reasoning behind the non-cooperative equilibrium and cooperative core distributions – both yielding $(1, 0, 0)$ – is as follows: *assuming* that the members of the assembly prefer something to nothing, whichever player B or C first receives from A some infinitesimal offer ϵ, he will hasten to accept it. In this game, then, a proper index should ascribe *full* power to A and *none* to B and C.

Such an index, however, would no longer be *a priori* – letting rational preferences slip into the analysis means that we are no longer measuring what voters *can* do but what (rational) voters *would* rather do. Nevertheless, the example points to a valid objection – *a priori* measures that focus only on outcomes about the success or failure of a ballot ignore outcomes that obtain between players. The next section presents a number of *a priori* measures that help elicit the amount of control A has over B and C without overemphasising it – after all, A might be a veto player but she is no *dictator*.

3 Indices of control, vulnerability and immunity

We already have a way of measuring the first element of the definition of power: a voter has the ability to bring about (an outcome) when he has a swing. What can be some relational outcomes that could be interesting in the context of voting? The example in the previous section can be of guidance. A is a veto player because she always has a swing. That is, she completely controls the failure (but not success) of the bill. But the reasoning underlying the critique says something more – not only has she power over her own success, she is also powerful with respect to the success of B and C. In other words, she *controls* B's and C's success as well as their own power over – ability to effect – the passage of the motion, i.e. their decisiveness. Similarly, if A controls B's and C's success, then B's and C's success is *vulnerable* to A. An immediate correlate, at least in the simple case of (zero-sum) yes-no voting,

[28] See also the response in Braham and Holler (2005a).

is that of immunity – if A's success is not vulnerable (to B and C), then we will say that it is *immune*. Let us take these ideas in turn and make them more precise.

3.1 Control

There seem to be at least two relational outcomes when it comes to control. A voter could control – have power over – either the *success* or *decisiveness* of another voter. Take the former. A straight-forward way of singling out those coalitions in which i has a swing and in which another voter $j \neq i$ is successful is by way of the *Controlled success index*, $cs_{i,-i}(v)$:[29]

$$cs_{i,-i} = \sum_{S \subseteq N} [v(S) - v(S \setminus \{i\})] \cdot \sum_{j \in S \setminus \{i\}} v(S)$$

The first argument is simply our formalisation of a swing. The second argument picks out the swing coalitions in which other voters are successful, i.e. coalitions in which the vote goes their way. Thus, $cs_{i,-i}(v)$ yields a number which falls between zero and the sum of all successful coalitions for other voters: $\sum\limits_{j \in S \setminus \{i\}} v(S)$.

To return to our previous example, let us see how much of B's C's success A controls. B and C each have two successful coalitions giving a total count of four. A has a swing in all four coalitions yielding $cs_{A,-A}(v) = 4$. On the other hand, B's C's respective measures are $cs_{B,-B}(v) = cs_{C,-C}(v) = 1$ (out of a total count of five).[30]

Now take decisiveness. We know what it means for a voter to have control as well as what it means to be decisive. Combining these two elements gives us the *Controlled decisiveness index*, $cd_{i,-i}(v)$:

$$cd_{i,-i}(v) = \sum_{S \subseteq N} [v(S) - v(S \setminus \{i\})] \cdot \sum_{j \in S \setminus \{i\}} [v(S) - v(S \setminus \{j\})]$$

In our previous example, for instance, B and C are decisive in one coalition each, with a total count of two, and A controls both: $cd_{A,-A}(v) = 2$. On the other hand, B and C control only one of all four decisive coalitions of other players: $cd_{B,-B}(v) = cd_{C,-C}(v) = 1$. This points to an interesting observation which starts

[29] Where $-i$ in the subscript indicates that control is taken over *all* voters other than i.

[30] All measures in this section are directly comparable to $\eta_i(v)$ (and not $\beta_i(v)$). This is simply more practical because the raw counts of various coalitions can be easily combined to give a series of informative indices, in the strict sense of the word, i.e. a number ranging between zero and one. In the case of $cs_{i,-i}(v)$, for instance, one could take the ratio between the actual controlled successful coalitions of a voter i ($cs_{i,-i}(v)$) and the potential controlled successful coalitions ($\sum\limits_{j \in S \setminus \{i\}} v(S)$).

to explain the critique launched by the proponents of strategic indices. To wit, as a veto player, A is not only completely decisive but has also *full* control over *both* the success and decisiveness of other players.[31]

3.2 Vulnerability

Just as a voter can have control over others' success and decisiveness, so can his success and decisiveness be vulnerable to (the control of) other players. A simple way of measuring the former is by asking how much of *each* successful coalition S of i is controlled by other voters belong to S. This yields the *Vulnerable success index*, $vs_{i,-i}(v)$:

$$vs_{i,-i}(v) = \sum_{\substack{S \subseteq N \\ i \in S}} v(S) \cdot \frac{\sum_{j \in S \setminus \{i\}} [v(S) - v(S \setminus \{j\})]}{|S| - 1}$$

a measure ranging between zero and the total number of i's successful coalitions. For instance, our veto player A is successful in three coalitions – only two of them are completely controlled by B or C, giving us $vs_{A,-A}(v) = 2$. B and C are each successful in two coalitions, one and a half of which are controlled by other players (namely, A): $vs_{B,-B}(v) = vs_{C,-C}(v) = \frac{3}{2}$.[32]

Analogously, the *Vulnerable decisiveness index*, $vd_{i,-i}(v)$, gives us the number of coalitions in which i is decisive and which are controlled by other voters:

$$vd_{i,-i}(v) = \sum_{S \subseteq N} [v(S) - v(S \setminus \{i\})] \cdot \frac{\sum_{j \in S \setminus \{i\}} [v(S) - v(S \setminus \{j\})]}{|S| - 1}$$

B and C, for example, are decisive in only one coalition each and that coalition is completely controlled by other voters (A): $vd_{B,-B}(v) = vd_{C,-C}(v) = 1$. A, on the other hand, is decisive in three coalitions, only two of which are controlled by either

[31] A feature shared by players who are *dictators*, i.e. who can single-handedly determine the outcome of a ballot. As both dictators and veto players are characterised by full control, this means that the defining property of the former should be sought elsewhere – namely, as we will see, in their complete immunity. In fact, the present relational *a priori* indices help delineate the differences between important types of players. Intuitively, dummy voters, i.e. voters who never have a swing, are completely *vulnerable*. Vetoers (and dictators) have full control, while dictators have full immunity. More precise statements and proofs of these results are available from the author upon request.

[32] Again, a useful index here would be the ratio between actual and potential vulnerable successful coalitions: $\frac{2}{3}$ for A, $\frac{3/2}{2} = \frac{3}{4}$ for B and C. That is, A has not only more control than B and C, she is also less vulnerable.

B or C: $vd_{A,-A}(v) = 2$.[33]

3.3 Immunity

Once we have an estimate of total success or decisiveness, on the one hand, and of vulnerable success or decisiveness, on the other, it is easy to measure immunity. Indeed, at least in the present framework, an immune outcome is simply an outcome that is not vulnerable. This gives us the *Immune success index*, $is_{i,-i}(v)$:

$$is_{i,-i}(v) = \underbrace{\sum_{\substack{S \subseteq N \\ i \in S}} v(S)}_{\text{total success}} - \underbrace{vs_{i,-i}(v)}_{\text{vulnerable success}}$$

and the *Immune decisiveness index*, $id_{i,-i}(v)$:

$$id_{i,-i}(v) = \underbrace{\eta_i(v)}_{\text{total decisiveness}} - \underbrace{vd_{i,-i}(v)}_{\text{vulnerable decisiveness}}$$

For instance, of A's three successful coalitions, two are vulnerable, yielding: $is_{A,-A}(v) = 1$. Of B's and C's two successful coalitions, on the other hand, three halves are vulnerable, yielding: $is_{B,-B}(v) = is_{C,-C}(v) = \frac{1}{2}$. Analogously, two out of A's three decisive coalitions are controlled by other voters, $id_{A,-A}(v) = 1$, while B's and C's single decisive coalition is completely vulnerable to A, $id_{B,-B}(v) = id_{C,-C}(v) = 0$. That is, while B and C have some ability to determine their and the ballot's success, that ability is void of any immunity to A's control.

A useful metric that we will revisit in the next section is a variant of $\beta_i(v)$ that looks not at total decisiveness but at immune decisiveness only. That is, at the proportion of decisive coalitions that are immune to the control of other voters. Because of its intimate relation to $\beta_i(v)$, let us call it the Immune Banzhaf index, $i\beta_i(v)$:

$$i\beta_i(v) = \frac{id_{i,-i}(v)}{2^{n-1}}$$

That is, $i\beta_i(v)$ is the proportion of i's actual immune decisive coalitions ($id_{i-,-i}(v)$) out of all potential decisive coalitions (2^{n-1}).

The point of the preceding analysis is that the US Supreme Court's quest for equal (*a priori*) voting power is not as straight-forward as it might seem. To wit, there is

[33] Again, the number of actual ($vd_{i,-i}(v)$) to potential ($\eta_i(v)$) vulnerable decisive coalitions is useful for comparison: $\frac{2}{3}$ for A, 1 for B and C. That is, while B and C have some ability to effect their success, that ability is completely vulnerable to other voters (A). A, on the other hand, has less than full vulnerability – she has, as we will see, some immunity.

no *single* measure of *a priori* power, such as decisiveness. Conferring various degrees of decisiveness to voters immediately leads to the attainment of other powers, such as control and immunity. These abilities are indeed related though not equivalent. The difficulty of evaluating fair representation based on equalising voting power stems from the fact that the type of power one chooses to equalise could lead to distributions pulling in different directions. To illustrate this point, let us conclude by looking at one prime application of *a priori* power theory – the analysis of representation in the European Union's Council of Ministers.

4 Power in the Council of Ministers

The decision-making process within the Council of Ministers between 1958 and 2002 can be analysed as a WVG.[34] During this period, the voting weights of the Council's members underwent a number of revisions to reflect enlargements of the European Union (Table 1). The goal of these enlargements, just as the goal of US reapportionment decisions, was to assign decision-making power to EU countries in proportion to their population. This, in effect, aimed to achieve a practicable equality of the worth of EU citizens' votes.

Past analyses in the *a priori* power literature, however, have produced a striking result – there has been throughout a consistent bias in favour of smaller sized states, i.e. a bias towards assigning them more power relative to their population share.[35] Tables 2 and 3 help us see this. Table 2 presents the ratio between a country's population and the population of the smallest sized state (Luxembourg).[36] A priori analyses of the power distribution in the Council is commonly based on some decisiveness measure. Table 3, for example, presents a similar ratio metric based on the Banzhaf index, $\beta_i(v)$ – that is, a country's $\beta_i(v)$ ratio is the ratio between its Banzhaf index and the Banzhaf index of the least powerful state (again, Luxembourg, or Belgium in 1958–72 when Luxembourg had zero decisiveness[37]).

[34] Ignoring, that is, the agenda-setting power of the Commission – see Garret and Tsebelis (1999: 301–305) and Napel and Widgrén (2011).

[35] Felsenthal and Machover (1997, 1998: 142–170) give a good summary plus references to some further studies.

[36] Thus, France's population ratio in 1958–72 is obtained as (in ′000): $\frac{\text{Pop}_{\text{FR}}}{\text{Pop}_{\text{LUX}}} = \frac{44{,}790}{310} = 144.4839$.

[37] Luxembourg's zero decisiveness is the famous 'blunder' (Hosli & Machover 2004: 503) of legislators in the early stages of the Council – its vote had technically no power to make *any* difference to the outcome of a ballot.

Table 1: Council of Ministers: Voting weights and quotas, 1958–2002

	State	1958–72†	1973–80†	1981–85†	1986–94†	1995–2002†
1	France	4	10	10	10	10
2	Germany	4	10	10	10	10
3	Italy	4	10	10	10	10
4	UK		10	10	10	10
5	Spain				8	8
6	Netherlands	2	5	5	5	5
7	Belgium	2	5	5	5	5
8	Greece			5	5	5
9	Portugal				5	5
10	Austria					4
11	Sweden					4
12	Denmark		3	3	3	3
13	Finland					3
14	Ireland		3	3	3	3
15	Luxembourg	1	2	2	2	2
	Total	**17**	**58**	**63**	**76**	**87**
	Quota	12	41	45	54	62

† These are the elements of the respective pure WVGs under the Treaty of Rome given that the Commission initiates the proposal. In case it does not, there is an additional condition of how many member states should support the decision – a number that has varied across the years.

Source: Felsenthal and Machover (1997).

Table 2: Council of Ministers: Population ratios, 1958–2002

	State	1958–72†	1973–80†	1981–85†	1986–94†	1995–2002†
1	France	144.4839	147.0822	148.3178	149.9351	145.3750
2	Germany	175.1290	175.5524	168.9315	164.8919	204.1000
3	Italy	158.1936	155.2068	154.7973	153.5676	143.2250
4	UK		158.6062	151.7452	153.4487	145.6500
5	Spain				104.4108	98.0250
6	Netherlands	36.0968	37.9632	38.9397	39.4135	38.6250
7	Belgium	29.1935	27.5921	26.9945	26.6919	25.3500
8	Greece			26.5781	27.0108	26.1500
9	Portugal				26.7486	24.7500
10	Austria					20.1250
11	Sweden					22.0750
12	Denmark		14.1841	14.0301	13.8351	13.0750
13	Finland					12.7750
14	Ireland		8.7422	9.4000	9.5730	8.9500
15	Luxembourg	1.0000	1.0000	1.0000	1.0000	1.0000

† The ratio is taken with respect to Luxembourg.

Source: Felsenthal and Machover (1998: 157) and own calculations.

Table 3: Council of Ministers: $\beta_i(v)$ ratios, 1958–2002

	State	1958–72†	1973–80	1981–85	1986–94	1995–2002
1	France	1.6667	10.6154	3.8445	7.1641	4.9301
2	Germany	1.6667	10.6154	3.8445	7.1641	4.9301
3	Italy	1.6667	10.6154	3.8445	7.1641	4.9301
4	UK		10.6154	3.8445	7.1641	4.9301
5	Spain				6.0615	4.0786
6	Netherlands	1.0000	5.8103	2.0000	3.7077	2.5939
7	Belgium	1.0000	5.8103	2.0000	3.7077	2.5939
8	Greece			2.0000	3.7077	2.5939
9	Portugal				3.7077	2.5939
10	Austria					2.1135
11	Sweden					2.1135
12	Denmark		4.2051	1.0000	2.5538	1.5852
13	Finland					1.5852
14	Ireland		4.2051	1.0000	2.5538	1.5852
15	Luxembourg	0.0000	1.0000	1.0000	1.0000	1.0000

† The ratio is taken with respect to Belgium.

Source: Own calculations.

Examining Table 2 and 3 indeed proves a stark contrast. To wit, given their population numbers, large states seem to have been truly under-represented based on their $\beta_i(v)$-measured decisiveness. Not only that but this under-representation seems to have *increased* over the years. For example, while Germany's population was consistently around 145 times that of Luxembourg's, its decisiveness had shrunk from nearly 11 times in 1973 to short of 5 times in 1995 that of the smallest state.

If decisiveness, however, is just one form of *a priori* power, what about the rest? To see how they can paint a different picture, let us look at one example – immunity. Table 5 provides the ratios of each member state's Immune Banzhaf index, $i\beta_i(v)$, to that of the smallest state (with positive $i\beta_i(v)$) while Table 4 recalculates the population ratios for better comparison.

What emerges from Tables 4 and 5 is a very different distribution. The first thing to note is Luxembourg's complete lack of immunity (in decisiveness) throughout the whole period. In other words, not only was it a *dummy* player, i.e. a player lacking any decisiveness, during 1958–1972 but also any ascribed decisiveness post-1972 had a twist – Luxembourg (and its voters) was given some positive ability to determine its own and the ballot's success but this was an ability entirely vulnerable to the control of others.

The same, in fact, was Denmark and Ireland's status in 1981–1985. More importantly, however, when one looks at $i\beta_i(v)$ instead of $\beta_i(v)$ – that is, at immune decisiveness instead of just decisiveness – the small-state bias starts to vanish around 1981 and is *completely reversed* in 1986 and 1995. To wit, what little disadvantage France, Germany, Italy and the UK might have had during 1958–1980, was in 1981 almost compensated and turned in 1986 into an advantage that was more than tripled in 1995.

Table 4: Council of Ministers: Amended population ratios, 1958–2002

	State	1958–72†	1973–80‡	1981–85ϒ	1986–94‡	1995–2002‡
1	France	4.9492	16.8244	5.5805	15.6623	16.2430
2	Germany	5.9989	20.0810	6.3560	17.2247	22.8045
3	Italy	5.4188	17.7537	5.8242	16.0418	16.0028
4	UK		18.1426	5.7094	16.0294	16.2737
5	Spain				10.9068	10.9525
6	Netherlands	1.2365	4.3425	1.4651	4.1172	4.3156
7	Belgium	1.0000	3.1562	1.0157	2.7883	2.8324
8	Greece			1.0000	2.8216	2.9218
9	Portugal				2.7942	2.7654
10	Austria					2.2486
11	Sweden					2.4665
12	Denmark		1.6225	—	1.4452	1.4609
13	Finland					1.4274
14	Ireland		1.0000	—	1.0000	1.0000
15	Luxembourg	—	—	—	—	—

† The ratio is taken with respect to Belgium.
‡ The ratio is taken with respect to Ireland.
ϒ The ratio is taken with respect to Greece.
Source: Felsenthal and Machover (1998: 157) and own calculations.

Table 5: Council of Ministers: $i\beta_i(v)$ ratios, 1958–2002

	State	1958–72†	1973–80‡	1981–85ϒ	1986–94‡	1995–2002‡
1	France	3.3376	10.2500	4.6780	19.4211	49.0000
2	Germany	3.3376	10.2500	4.6780	19.4211	49.0000
3	Italy	3.3376	10.2500	4.6780	19.4211	49.0000
4	UK		10.2500	4.6780	19.4211	49.0000
5	Spain				13.7895	33.8571
6	Netherlands	1.0000	2.5000	1.0000	3.2105	9.2857
7	Belgium	1.0000	2.5000	1.0000	3.2105	9.2857
8	Greece			1.0000	3.2105	9.2857
9	Portugal				3.2105	9.2857
10	Austria					4.5714
11	Sweden					4.5714
12	Denmark		1.0000	0.0000	1.0000	1.0000
13	Finland					1.0000
14	Ireland		1.0000	0.0000	1.0000	1.0000
15	Luxembourg	0.0000	0.0000	0.0000	0.0000	0.0000

† The ratio is taken with respect to Belgium.
‡ The ratio is taken with respect to Ireland.
ϒ The ratio is taken with respect to Greece.
Source: Own calculations.

Now, large states might have had much less decisiveness and much more immunity relative to their population during the period but what is the significance of this? Put differently, what is the fair criteria against which an institution should – *a priori* – distribute abilities such as decisiveness and immunity? Here, it becomes imperative to distinguish between *a priori* descriptive and normative analysis. It is one thing to delineate and measure in some intuitive or *methodologically* debatable way the kinds of abilities decision rules grant but a much different thing to evaluate them normatively. What is the fair distribution – yielding fair representation – of decisiveness and immunity?[38] Should maximal population proportionality be strived for across all such voting powers, and if that is impossible, what compromises should be made? Are success and decisiveness more imperative than control and immunity? Should designers of voting bodies take them all into account? Having once recognised the plurality of *a priori* powers present in a voting environment, these are the kinds of questions that a normative analysis needs to address.

[38] Respectively, control, vulnerability, success, etc.

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